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NOTE ON (A,m)-ISOMETRIC OPERATORS IN SEMI-HILBERTIAN SPACES

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Abstract: Given a bounded positive linear operator A on a Hilbert space $\mathscr{H}$ we consider the semi-Hilbertian space $(\mathscr{H}, \langle\, |\, \rangle_A)$, where $\langle \xi \mid \eta \rangle_A := \langle A\xi \mid \eta \rangle$. In this note we study further properties of the class of (A,m)-isometric operators on a semi Hilbertian space $\mathscr{H}$ with inner product $\langle\, |\, \rangle_A$. A Hilbert space operator $T \in \mathscr{B}[\mathscr{H}]$ is (A,m)-isometry for some $A \in \mathscr{B}[\mathscr{H}]^+$ and integer $m \geq 1$ ([19]) if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^{*k} A T^k = 0.$$

Keywords: Hilbert space; Hilbert space operator; isometric operators.

1. Introduction

Along this note $\mathscr{H}$ denotes a complex Hilbert space with inner product $\langle .|. \rangle$. $\mathscr{B}[\mathscr{H}]$ is the algebra of all bounded linear operators on $\mathscr{H}$, $\mathscr{B}[\mathscr{H}]^+$ is the cone of positive (semi-definite) operators of $\mathscr{B}[\mathscr{H}]$, i.e.,

$$\mathscr{B}[\mathscr{H}]^+ := \{T \in \mathscr{B}[\mathscr{H}] \;\; \langle T\xi | \xi \rangle \geq 0 \;\; \forall \xi \in \mathscr{H} \}$$

and $\mathscr{B}(\mathscr{H})_{cr}$ is the subset of $\mathscr{B}[\mathscr{H}]$ of all operators with closed range. For every $T \in \mathscr{B}[\mathscr{H}]$, $\mathscr{N}(T), \mathscr{R}(T)$ and $\overline{\mathscr{R}(T)}$ stand for, respectively, the null space, the range and the closure of the range of T. its adjoint operator by T^*. In addition, if $T_1, T_2 \in \mathscr{B}[\mathscr{H}]$ then $T_1 \geq T_2$ means that $T_1 - T_2 \in \mathscr{B}[\mathscr{H}]^+$. Given a closed subspace $\mathscr{S}$ of $\mathscr{H}$, $P_{\mathscr{S}}$ denotes the orthogonal projection onto $\mathscr{S}$. On the other hand, $T^\dagger$ stands for the Moore-Penrose inverse of $T \in \mathscr{B}[\mathscr{H}]$.

Given $A \in \mathscr{B}[\mathscr{H}]^+$, the functional

$$\langle\, |\, \rangle_A : \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{C}, \; \langle \xi \mid \eta \rangle_A = \langle A\xi \mid \eta \rangle$$

is a semi-inner product on $\mathscr{H}$. By $\|.\|_A$ we denote the seminorm induced by $\langle\, |\, \rangle_A$, i.e.,$\|\xi\|_A = \langle \xi | \, \xi \rangle_A^{\frac{1}{2}}$. Observe that $\|\xi\|_A = 0$ if and only if $\xi \in \mathscr{N}(A)$. Then $\|.\|_A$ is a norm if and only if A is an injective operator,and the seminormed space $(\mathscr{H}, \|.\|_A)$ is complete if and only if $\mathscr{R}(A)$ is closed. Moreover, $\langle\, |\, \rangle_A$ induces a semi-norm on a certain subspace of $\mathscr{B}[\mathscr{H}]$, namely, on the subspace

$$\{ T \in \mathscr{B}[\mathscr{H}] / \; \exists \, c > 0 : \; \|T\xi\|_A \leq c\|\xi\|_A \;\; \forall \, \xi \in \mathscr{H} \}.$$

In such case it holds

$$\begin{aligned}\|T\|_A &= \sup_{\substack{\xi \in \overline{\mathscr{R}(A)} \\ \xi \neq 0}} \frac{\|T\xi\|_A}{\|\xi\|_A} = \sup_{\|\xi\|_A \leq 1} \|T\xi\|_A = \sup\{\|T\xi\|_A \,:\, \|\xi\|_A = 1\,\} \\ &= \inf\{c > 0 : \|T\xi\|_A \leq c\|\xi\|_A,\ \xi \in \mathscr{H}\,\} < \infty.\end{aligned}$$

Moreover

$$\|T\|_A = \sup\{\langle T\xi \mid \eta\rangle_A;\, \xi, \eta \in \mathscr{H}, : \|\xi\| \leq 1, \|\eta\| \leq 1\,\}.$$

For $\xi, \eta \in \mathscr{H}$, we say that ξ and η are A-orthogonal if $\langle \xi \mid \eta\rangle_A = 0$. Define

$$\mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H}) := \{\, T \in \mathscr{B}(\mathscr{H}) :\ \|T\xi\|_A \leq c\|\xi\|_A \ \text{ for every } \ \xi \in \mathscr{H}\,\}$$

It is easy to see that $\mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is a subspace of $\mathscr{B}(\mathscr{H})$.

Definition 1.1. ([5]) For $T \in \mathscr{B}(\mathscr{H})$, an operator $S \in \mathscr{B}(\mathscr{H})$ is called an A-adjoint of T if for every $\xi, \eta \in \mathscr{H}$

$$\langle T\xi \mid \eta\rangle_A = \langle \xi \mid S\eta\rangle_A,$$

i.e., $AS = T^*A$; we say that T is A-self-adjoint if $AT = T^*A$.

or which is equivalent,if S is a solution of the equation $AX = T^*A$.

An operator acting on a Hilbert space $\mathscr{H}$ is called m-isometric for some integer $m \geq 1$ if

$$T^{*m}T^m - \binom{m}{1}T^{*m-1}T^{m-1} + ... + (-1)^{m-1}\binom{m}{m-1}T^*T + (-1)^m I = 0 \tag{1.1}$$

where $\binom{m}{k}$ be the binomial coefficient. A simple manipulation proves that (1.1) is equivalent to

$$\sum_{k=0}^{m} (-1)^{m-k}\binom{m}{k}\|T^k x\|^2 = 0, \ \text{ for all } x \in \mathscr{H} \tag{1.2}$$

Evidently, an isometric operator (i.e., a 1-isometric operator) is m-isometric for all integers $m \geq 1$. Indeed the class of m-isometric operators is a generalization of the class of isometric operators and a detailed study of this class and in particular 2-isometric operators on a Hilbert space has been the object of some intensive study, especially by J.Agler and M. Stankus in [1, 2, 3], also by S.Richter [21] Shimorin [22] ,S.M. Patel [18] and B.P.Duggal in [13, 14]. m-Isometries are not only a natural extension of an isometry, but they are also important in the study of Dirichlet operators and some other classes of operators.

A generalization of m-isometries to operators on general Banach spaces has been presented by several authors in the last years. Botelho [11] and Sid Ahmed [20] discuss operators defined via (1.2) on (complex) Banach spaces. Bayart introduces in [8] the notion of (m, p)-isometries on general (real or complex) Banach spaces. An operator

$T \in \mathscr{B}[X]$ on a Banach space X is called an (m,p)-isometry if there exists an integer $m \geq 1$ and a $p \in [1,\infty)$, with

$$\forall x \in X, \ \sum_{k=0}^{m}(-1)^k \binom{m}{k} \|T^{m-k}x\|^p = 0 \tag{1.3}$$

It is easy to see that, if $X = \mathscr{H}$ is a Hilbert space and $p = 2$, this definition coincides with the original definition (1.1) of m-isometries. In [16] the authors took off the restriction $p \geq 1$ and defined (m,p)-isometries for all $p > 0$. They studied when an (m,p)-isometry is an (μ,q)-isometry for some pair (μ,q). In particular, for any positive real number p they gave an example of an operator T that is a $(2,p)$-isometry, but is not a $(2,q)$-isometry for any q different from p. In [9] and [10] it is proven that the powers on an m-isometry are m-isometries and some products of m-isometries are again m-isometries. For any $T \in \mathscr{B}(\mathscr{H})$ we set

$$\beta_m(T) := \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}T^{*j}T^j. \tag{1.4}$$

Let $A \in \mathscr{B}[\mathscr{H}]^+$ and let m be a positive integer. An operator $T \in \mathscr{B}[\mathscr{H}]$ is said to be an (A,m)-isometry if and only if

$$\beta_m^A(T) := \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}AT^k = 0 \tag{1.5}$$

or equivalently if

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|T^k x\|_A^2 = 0 \tag{1.6}$$

for all $x \in \mathscr{H}$.

In particular, if T is a $(A,2)$-isometry or a $(A,3)$-isometry, then it must satisfy the operator equation

$$T^{*2}AT^2 - 2T^*AT + A = 0 \tag{1.7}$$

or

$$T^{*3}AT^3 - 3T^{*3}AT^3 + 3T^*AT - A = 0 \tag{1.8}$$

respectively.

Every $(A,1)$-isometry or A-isometry(that is T satisfying $T^*AT = A$) is an (A,m)-isometry. It follows from (1.7) and (1.8) that every $(A,2)$-isometry is a $(A,3)$-isometry. More generally it is true that an (A,m)-isometry is also an (A,n)-isometry for all $n \geq m$, cf. [19]. The class of (A,m)-isometries has been introduced by Sid Ahmed and Saddi [19].

In recent years, several results covering some classes of operators on a complex Hilbert space $(\mathscr{H}, \langle .\,|.\,\rangle)$ are extended to $(\mathscr{H}, \langle .\,|.\,\rangle_A)$. [6, 7, 15]

2. Main results

In this section, we study further properties for some (A,m)-isometric operators.

The following theorem gives a characterization of $(A,3)$-isometric operators.

Theorem 2.1. *Let $T \in \mathscr{B}(\mathscr{H})$. Then T is an $(A,3)$-isometric operator if and only if T satisfies*

$$T^{*n}AT^n = A + n\Psi_1(T^*,A,T) + n^2\Psi_2(T^*,A,T) \tag{2.1}$$

for $n = 0,1,2,\cdots$, where

$$\Psi_2(T^*,A,T) = \frac{1}{2}\Big(T^{*2}AT^2 - 2T^*AT + A\Big) \tag{2.2}$$

and

$$\Psi_1(T^*,A,T) = \frac{1}{2}\Big(-T^{*2}AT^2 + 4T^*AT - 3A\Big) \tag{2.3}$$

Proof. We prove the if part of the theorem. Assume that T satisfies (2.1). For $n = 3$ we obtain

$$\begin{aligned} T^{*3}AT^3 &= A + 3\Psi_1(T^*,A,T) + 9\Psi_2(T^*,A,T) \\ &= A + \frac{3}{2}\Big(-T^{*2}AT^3 + 4T^*AT - 3A\Big) + \frac{9}{2}\Big(T^{*2}AT^2 - 2T^*AT + A\Big) \\ &= A - 3T^{*2}AT^2 - 3T^*AT. \end{aligned}$$

Hence, we have

$$T^{*3}AT^3 - 3T^{*2}AT^2 + 3T^*AT - A = 0,$$

and so that, T is an $(A,3)$-isometry.

We prove the only if part. Assume that T is an $(3,A)$-isometry. We prove (2.1 by mathematical induction. For $n = 1$ it is true. Assume that (2.1) is true for n and prove it for $n+1$. Indeed, we have

$$\begin{aligned} T^{*n+1}AT^{n+1} &= T^*(T^{*n}AT^n)T \\ &= T^*\Big(A + n\Psi_1(T^*,A,T) + n^2\Psi_2(T^*x,A,T)\Big)T \\ &= T^*AT + \frac{n}{2}\Big(-T^{*3}AT^3 + 4T^{*2}AT^2 - 3T^*AT\Big) \\ &\quad + \frac{n^2}{2}\Big(T^{*3}AT^3 - 2T^{*2}AT^2 + T^*AT\Big) \\ &= \Big(\frac{n^2-n}{2}\Big)T^{*3}AT^3 - (n^2-2n)T^{*2}AT^2 + \Big(\frac{n^2-3n+2}{2}\Big)T^*AT. \end{aligned}$$

Now, using the fact that T is an $(A,3)$-isometry we can obtained

$$\begin{aligned} T^{*n+1}AT^{n+1} &= \left(\frac{n^2-n}{2}\right)\left(A+3T^{*2}AT^2-3T^*AT\right)+-(n^2-2n)T^{*2}AT^2 \\ &\quad +\left(\frac{n^2-3n+2}{2}\right)T^*AT \\ &= \left(\frac{n^2+n}{2}\right)T*2AT^2+\left(\frac{-2n^2+2}{2}\right)T^*AT+\left(\frac{n^2-n}{2}\right)A \\ &\quad =\left(\frac{n^2+n}{2}\right)\left(A+2\Psi_1(T^*,A,T)+4\Psi_2(T^*,A,T)\right) \\ &\quad +\left(\frac{-2n^2+2}{2}\right)\left(A+\Psi_1(T^*,A,T)+\Psi_2(T^*,A,T)\right)+\left(\frac{n^2-n}{2}\right)A \\ &= A+(n+1)\Psi_1(T^*,A,T)+(n+1)^2\Psi(T^*,A,T). \end{aligned}$$

□

Theorem 2.2. *Let $T\in\mathscr{B}(\mathscr{H})$ and $S\in\mathscr{B}(\mathscr{H})$ for which $TS=ST$. The following properties hold*

(1) *If S is an A-isometry, then*

$$\beta_m^A(TS)=\beta_m^A(T). \tag{2.4}$$

(2) *If S is an $(A,2)$-isometry, then*

$$\beta_{m+1}^A(TS)=(m+1)S^*T^*\beta_m^A(T)TS-(m+1)T^*\beta_m^A(T)T+\beta_{m+1}^A(T). \tag{2.5}$$

(3) *If S is an $((A,3)$-isometry, then*

$$\begin{aligned} \beta_{m+1}^A(TS) &= \beta_{m+2}^A(T)+T^*\beta_m^{\Psi_1(S^*,A,S)}(T)T+(m+2)(m+1)T^{*2}\beta_m^{\Psi_2(S^*,A,S)}(T)T^2 \\ &\quad +(m+2)T^*\beta_{m+1}^{\Psi_2(S^*,A,S)}(T)T. \end{aligned} \tag{2.6}$$

Proof. (1) Assume that S is an A-isometry, then we have $S^{*k}AS^k=A\ \forall\ k=0,1,2,\cdots$ and it follows that

$$\begin{aligned} \beta_m^A(TS) &= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}(TS)^{*k}A(TS)^k \\ &= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}T^{*k}S^{*k}AT^kS^kx) \\ &= \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}T^{*k}(S^{*k}AS^k)T^k \\ &= \beta_m^A(T) \\ &= 0. \end{aligned}$$

(2) Assume that S is an $(A,2)$-isometry. Then we have by using [19, Lemma3.4] that

$S^{*k}AS^k = kS^*AS + (1-k)A$ for $k = 0,1,\cdots$. A simple calculation shows that

$$\begin{aligned}
\beta^A_{m+1}(TS) &= \sum_{0\le k\le m+1}(-1)^{m+1-k}\binom{m+1}{k}(TS)^{*k}A(TS)^k \\
&= \sum_{0\le k\le m+1}(-1)^{m+1-k}\binom{m+1}{k}T^{*k}\left(S^{*k}AS^k\right)T^k \\
&= \sum_{0\le k\le m+1}(-1)^{m+1-k}\binom{m+1}{k}T^{*k}\left(kS^*AS+(1-k)A\right)T^k \\
&= S^*\left(\sum_{1\le k\le m+1}(-1)^{m+1-k}k\binom{m+1}{k}T^{*k}AT^k\right)S + \sum_{0\le k\le m+1}(-1)^{m+1-k}\binom{m+1}{k}(-k+1)T^{*k}A \\
&= (m+1)S^*\left(\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}T^{*k+1}AT^{k+1}\right)S - (m+1)T^*\beta^A_m(T)T + \beta^A_{m+1}(T) \\
&= (m+1)S^*T^*\beta^A_m(T)TS - (m+1)T^*\beta^A_m(T)T + \beta^A_{m+1}(T) \\
&= 0
\end{aligned}$$

(3) Assume that S is an $(A,3)$-isometry and $TS = ST$. In view of Theorem 2.1 we have that

$$\begin{aligned}
\beta^A_{m+2}(TS) &= \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}(ST)^{*k}A(ST)^k \\
&= \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}T^{*k}\left(S^{*k}AS^k\right)T^k \\
&= \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}T^{*k}\left(A+k\Psi_1(S^*,A,S)+k^2\Psi_2(S^*,A,S)\right)T^k \\
&= \Bigg\{\sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}(T)^{*k}AT^k + \\
&\quad \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}k(T)^{*k}\Psi_1(S^*,A,S)T^k + \\
&\quad \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}k^2T^{*k}\Psi_2(S^*,A,S)T^k\Bigg\} \\
&= \beta^A_{m+2}(T) + T^*\beta^{\Psi_1(S^*,A,S)}_m(T)T + (m+2)(m+1)T^{*2}\beta^{\Psi_2(S^*,A,S)}_m(T)T^2 \\
&\quad +(m+2)T^*\beta^{\Psi_2(S^*,A,S)}_{m+1}(T)T.
\end{aligned}$$

□

The proof of the following corollary is an immediate consequence of Theorem 2.2

Corollary 2.1. *Let $T,S \in \mathcal{B}[\mathcal{H}]$ such that $TS = ST$. If T is an (A,m)-isometry and S is an (A,k)-isometry for $k = 1,2,3$, then TS is an $(A,m+k-1)$-isometry for $k = 1,2,3$.*

Let $\mathcal{H}\overline{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform crose-norm, of the algebraic tensor product $\mathcal{H}\otimes\mathcal{H}$ of $\mathcal{H}$ with $\mathcal{H}$. Given non-zero $T,S \in \mathcal{B}(\mathcal{H})$, let $T\otimes S\in\mathscr{B}(\mathscr{H}\overline{\otimes}\mathscr{H})$ denote the tensor product on the Hilbert space $\mathscr{H}\overline{\otimes}\mathscr{H}$, when $T\otimes S$ is defined as follows

$$\langle T\otimes S(\xi_1\otimes\eta_1)|\,(\xi_2\otimes\eta_2)\rangle=\langle T\xi_1|\,\xi_2\rangle\langle S\eta_1|\,\eta_2\rangle.$$

The operation of taking tensor products $T\otimes S$ preserves many properties of $T,S\in\mathscr{B}(\mathscr{H})$, but by no means all of them. Thus, whereas $T\otimes S$ is normal if and only if T and S are normal [17], there exist paranormal operators T and S such that $T\otimes S$ is not paranormal [4]. In [12], Duggal showed that if for non-zero $T,S\in\mathscr{B}(\mathscr{H}),T\otimes S$ is p-hyponormal if and only if T and S are p-hyponormal.

Proposition 2.1. *Let $T,S\in\mathscr{B}[\mathscr{H}]$ and $A,B\in\mathscr{B}[\mathscr{H}]^+$. Then the following hold:*

(1) T is an (A,m)-isometry if and only if $T\otimes I$ is an $(A\otimes B,m)$-isometry.

(2) S is an (B,m)-isometry if and only if $I\otimes S$ is an $(A\otimes B,m)$-isometry.

Proof. The proof follows from the following identities.

$$\beta_m^{A\otimes B}(T\otimes I)=\beta_m^A(T)\otimes B$$

and

$$\beta_m^{A\otimes B}(I\otimes S)=A\otimes\beta_m^B(S).$$

Theorem 2.3. *Let $T,S\in\mathscr{B}[\mathscr{H}]$ and $A,B\in\mathscr{B}[\mathscr{H}]^+$. If T is an (A,m)-isometry and S is an (B,k)-isometry for $k=1,2,3$. Then $T\otimes S$ is an $(A\otimes B,m+k-1)$-isometry for $k=1,2,3$.*

Proof. Two proofs for this theorem will be given.

The First Proof. For $k=1$, then

$$\begin{aligned}\beta_m^{A\otimes B}(T\otimes S)&=\sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}(T\otimes S)^{*j}(A\otimes B)(T\otimes S)^j\\&=\sum_{0\leq j\leq m}(-1)^{m-i}\binom{m}{j}(T^{*j}AT^j)\otimes(S^{*j}BS^j)\\&=\sum_{0\leq j\leq m}(-1)^{m-i}\binom{m}{j}(T^{*j}AT^i)\otimes B\\&=\beta_m^A(T)\otimes B\\&=0.\end{aligned}$$

For $k = 2$. We have

$$
\begin{aligned}
\beta_{m+1}^{A\otimes B}(T\otimes S) &= \sum_{0\leq j\leq m+1} (-1)^{m+1-i}\binom{m+1}{j}(T\otimes S)^{*j}(A\otimes B)(T\otimes S)^{j} \\
&= \sum_{0\leq j\leq m} (-1)^{m-j}\binom{m}{j}(T^{*j}AT^{j})\otimes(S^{*j}BS^{j}) \\
&= \sum_{0\leq j\leq m} (-1)^{m-i}\binom{m}{j}(T^{*j}AT^{j})\otimes B \\
&= \sum_{0\leq j\leq m+1} (-1)^{m+1-j}\binom{m+1}{j}\Big(T^{*j}AT^{j}\otimes\big(jS^{*}BS+(1-j)B\big)\Big) \\
&= \sum_{0\leq j\leq m+1} (-1)^{m+1-j}\binom{m+1}{j} jT^{*j}AT^{j}\otimes S^{*}BS \\
&\quad - \sum_{0\leq j\leq m+1} (-1)^{m+1-j}\binom{m+1}{j} jT^{*j}AT^{j}\otimes B \\
&\quad + \sum_{0\leq j\leq m+1} (-1)^{m+1-j}\binom{m+1}{j} jT^{*j}AT^{j}\otimes B \\
| &= (m+1)\beta_{m}^{A}(T)\otimes S^{*}BS-(m+1)\beta_{m}^{A}(T)\otimes B+\beta_{m+1}^{A}(T)\otimes B \\
&= 0.
\end{aligned}
$$

For $k = 3$ we have

$$
\begin{aligned}
& \beta_{m+2}^{A\otimes B}(T\otimes S) \\
= & \sum_{0\leq j\leq m+2} (-1)^{m+2-j}\binom{m+2}{j}(T\otimes S)^{*j}(A\otimes B)(T\otimes S)^{j} \\
= & \sum_{0\leq j\leq m+2} (-1)^{m+2-j}\binom{m+2}{j}(T^{*j}AT^{j})\otimes(S^{*j}BS^{j}) \\
= & \sum_{0\leq j\leq m+2} (-1)^{m+2-j}\binom{m+2}{j}(T^{*j}AT^{j})\otimes\big(B+j\Psi_1(S^{*},B,S)+j^{2}\Psi_2(S^{*},B,ST)\big)\Big) \\
= & \sum_{0\leq j\leq m+2} (-1)^{m+2-j}\binom{m+2}{j}\Big((T^{*j}AT^{j}\otimes B \\
& + \sum_{0\leq j\leq m+2} (-1)^{m+2-j}\binom{m+2}{j} j\big((T^{*j}AT^{j})\otimes\Psi_1(S^{*},B,S) \\
& + \sum_{0\leq j\leq m+2} (-1)^{m+2-j}\binom{m+2}{j} j^{2}\big((T^{*j}AT^{j})\otimes\Psi_2(S^{*},B,S) \\
= & \beta_{m+2}(T)\otimes B+T^{*}\beta_{m}^{A}(T)T\otimes\Psi_1(S^{*},B,S)+(m+2)T^{*}\beta_{m+1}^{A}(T)T\otimes\Psi_2(S^{*},B,S) \\
& +(m+2)(m+1)T^{*2}\beta_{m}^{A}(T)T^{2}\otimes\Psi_2(S^{*},B,S) \\
= & 0.
\end{aligned}
$$

3. Conclusions

This concludes the first proof.

The Second Proof. By observing that

$$T\otimes S=(T\otimes I)(I\otimes S)=(I\otimes S)(T\otimes I).$$

Since T is an (A,m)-isometry and S is an (B,k)-isometry for $k=1,2,3$, it follows in view of Proposition 2.1 that $T\otimes I$ is an $(A\otimes B,m)$ and $I\otimes S$ is an $(A\otimes B,k)$-isometry for $k=1,2,3$. By applying Theorem 2.2, we deduce that $T\otimes S$ is an $(A\otimes B,m+k-1)$-isometry.

Theorem 2.4. *Let $T,S\in\mathscr{B}[\mathscr{H}]$ and $A,B\in\mathscr{B}[\mathscr{H}]^{+}$. If T is an (A,m)-isometry and S is an (B,n)-isometry, then $T\oplus S$ is an $(A\oplus B,p)$-isometry, where $p=\max\{m,n\}$.*

Proof.

$$\begin{aligned}
\beta_p^{A\oplus B}(T\oplus S) &= \sum_{0\le k\le p}(-1)^{p-k}\binom{p}{k}(T\oplus S)^{*k}(A\oplus B)(T\oplus S)^{k}\\
&= \sum_{0\le k\le p}(-1)^{p-k}\binom{m}{k}\left(T^{*k}AT^{k}\oplus S^{*k}BS^{k}\right)\\
&= \left(\sum_{0\le k\le p}(-1)^{p-k}\binom{p}{k}T^{*k}AT^{k}\right)\oplus\left(\sum_{0\le k\le p}(-1)^{p-k}\binom{p}{k}S^{*k}BS^{k}\right)\\
&= \beta_p^{A}(T)\oplus\beta_p^{B}(S)\\
&= 0.
\end{aligned}$$

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